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Geometric methods in physics: spinors and twistors

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Abstract

Roger Penrose developed the Twistor Theory to unify Quantum Field Theory with General Relativity. In this new way of thinking, space-time is viewed as emerging from a more fundamental complex projective space, the twistor space. These spaces are connected by a straightforward equation, from which one can obtain space-time conformal structure. Finally, the deformation of twistor space is used to construct the Ricci-flat metric of the associated space-time.

Key-words: Twistors. Non-linear Graviton. General Relativity.

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1 Introduction

It is well known that the two main branches of Theoretical Physics that give the most modern understanding of Nature rest upon two formalisms: General Relativity (GR) and Quantum Field Theory (QFT). Since their developments, several geometric methods have increasingly occupied the center of attention in Theoretical Physics. Among them, what stood out the most was the concept of *spinors*, fundamental building blocks of a relativistic QFT. A work developed by Witten[1] introduced an application of spinors in space-time, but in a somewhat different way from the one used in QFT, leading to the use of this tool in GR.

However, it is widely accepted that the "big picture" is incomplete, lacking an entirely consistent framework encompassing these two successful physics branches. Aiming at this problem, Penrose in 1967[2] introduces, using spinors, the concept of *twistors*. Twistors are elements of a complex projective space related to the causal structure of space-time through one central equation, known as the incidence relation. This theory changes the typical way of thinking about relativity. Space-time is no longer viewed as fundamental but as emerging from the twistor space. Therefore, it introduced a new way of looking at and thinking about GR, which can be used to further developments of the theory of quantum gravity.

From Twistor Theory, innumerable progress in both mathematics and physics was made. For example, in the theory of integrable equations, one can construct a function, using coordinates of a twistor, that automatically satisfies the wave equation in space-time[3,4]. In the context of quantum gravity, twistor theory gave a new way of quantizing gravity[5]. Besides, the existence of twistors in some space-time $\mathscr{M}$ imposes certain conditions to the curvature of this space-time (see section 7), known as the non-linear graviton construction.

In more recent years, twistor theory has been used in the context of string theory[6] and in the problem of spontaneous symmetry breaking[7]. The last great use of twistor is to create a mechanism that explains the quantum state reduction[8], using some ideas of the non-linear graviton construction. Therefore, all the researches cited above reinforce the role of twistors in modern physics, although they are beyond the scope of this project.

In this work, the basic geometric and algebraic properties of twistors are presented, emphasizing the relationship between the twistor space and space-time. Then, this view is extended to complexified Minkowski space-time and its relation with complex projective space. Lastly, the general ideas of the non-linear graviton construction are presented, with calculations of the metric and the curvature tensors.

2 Spinor Theory

The basic building blocks of Twistor Theory are the so-called *spinors*. They arise most naturally in studying relativistic quantum field theory, for instance, in Dirac's and Weyl's equations. However, the helpful picture of spinors in relativity (called 2-spinors in some references) is drastically different from the one adopted in quantum theory. Therefore, this section presents the main ideas behind this theory without going deep into the differential geometry aspects of the theory.

Let $\mathscr{M}$ be the Minkowski space-time. The special covariant principle implies that the isometry group of such space-time acts naturally on the states of a physical theory defined on it. In the context of quantum field theory, this leads to the study of the unitary representations of the Poincaré Group (isometry group of $\mathscr{M}$). Since state vectors that differ by a phase factor represent the same physical state in a Hilbert space, representations "up to phase" of the Poincaré group are allowed. These representations are equivalent to representations "up to sign" of the covering group of the Poincaré group, which is isomorphic to $\mathrm{ISL}(2,\mathbb{C})^{\dagger}$, the group of all translations and linear maps with unit determinant acting on a two-dimensional complex vector space. In this context, **spinors** are defined as the carriers of the true representation of $\mathrm{SL}(2,\mathbb{C})$. Therefore, there is a way to obtain tensor fields (e.g., the metric) from spinor fields[9,10].

The essential fact about 2-spinors calculus is based on the local isomorphism between the Lorentz group and $\mathrm{SL}(2,\mathbb{C})$. This morphism can be established in the following way. Let u^a be a vector in $\mathscr{M}$. Let u^μ be its components in a Minkowski basis. Consider, now, the scheme

$$u^\mu = (u^0, u^1, u^2, u^3) \leftrightarrow u^{\Sigma\Sigma'} = \begin{pmatrix} u^{00'} & u^{01'} \\ u^{10'} & u^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u^0 + u^3 & u^1 + iu^2 \\ u^1 - iu^2 & u^0 - u^3 \end{pmatrix},$$

where $u^{\Sigma\Sigma'}$ are the components of a 2-spinor $u^{AA'}$ associated with $u^a \in \mathscr{M}$. It follows from this scheme that when u^μ undergo a restricted Lorentz transformation L, the $u^{\Sigma\Sigma'}$ undergo

$$u^{\Sigma\Sigma'} \to A \begin{pmatrix} u^0 + u^3 & u^1 + iu^2 \\ u^1 - iu^2 & u^0 - u^3 \end{pmatrix} A^{\dagger}, \tag{1}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $det(A) = 1$ and a, b, c and d are complex. This establishes the two-fold covering of the Lorentz group by $\mathrm{SL}(2,\mathbb{C})$, since both A and $-A$ give rise to the same Lorentz transformation of u^μ by (1)[9,10]. This scheme is the definition of a spinor.

Primed and unprimed indices must be treated differently and are related under complex conjugation.

†The group $\mathrm{SL}(2,\mathbb{C})$ is composed by all special linear maps in $\mathbb{C}^2$. The group $\mathrm{ISL}(2,\mathbb{C})$ contains, in addition, the translation maps.

From (1) it follows that each abstract tensor index (four-dimensional) is to be equated with a pair of spinors indices (two-dimensional), one primed and one unprimed. Thus, tensor indices a,b, may be expressed by $a = AA'$, $b = BB'$, and so forth.

Let ϵ_{AB} be an antisymmetric and nondegenerated (i.e., $\epsilon_{AB}\psi^B \neq 0$, unless $\psi^B = 0$) tensor. One can use ϵ_{AB} to map spinors into dual spinors via $\psi^A \to \epsilon_{AB}\psi^A$. However, since ϵ_{AB} is antisymmetric, it makes a difference which index of ϵ_{AB} is contracted with the spinor index of ψ^A. We follow the standard convention of contraction over the first index of ϵ_{AB} and over the second index of ϵ^{AB} (which satisfies $\epsilon^{AB}\epsilon_{BC} = -\delta^A{}_C$). Therefore, $\psi_B = \epsilon_{AB}\psi^A$ and $\psi^B = \epsilon^{BA}\psi_A$; as a consequence, we have

$$\psi^A\mu_A = -\psi_B\mu^B \Rightarrow \psi^A\psi_A = 0, \forall\ \psi^A.$$

It follows from the last result above that if $\xi^A\psi_A = 0$, then ξ^A is a scalar multiple of ψ^A or one of them is zero.

A real null vector u^a ($u_a u^a = 0$) has a spinor form

$$u^{AA'} = \pm\xi^A\bar{\xi}^{A'}\ . \tag{2}$$

The plus sign occurs if u^a is future-pointing and the minus if it is past-pointing. However, the vector u^a given by (2) corresponds to a family of spinors given by all spinors of the form $e^{i\theta}\xi^A, \forall\theta \in \mathbb{R}$. Therefore, obtaining a more complete tensorial realization of a spinor is necessary. This geometrical pictured is called *null flag*.

The construction of the null flag goes as follows. Let k^A be a spinor and let $K^a = k^A\bar{k}^{A'}$ be a null vector. Let F^{AB} be a real antisymmetric bivector given by

$$F^{ab} = k^A k^B \epsilon^{A'B'} + \epsilon^{AB}\bar{k}^{A'}\bar{k}^{B'}\ . \tag{3}$$

Let τ^A be a spinor such that $k_A\tau^A = 1$[†]. Refs[9,10] argue that using these two spinors, it follows that $\epsilon^{AB} = k^A\tau^B - \tau^A k^B$. Therefore, one can write $F^{ab} = K^a L^b - L^a K^b$, where the vector $L^a = k^A\bar{\tau}^{A'} + \tau^A\bar{k}^{A'}$ is real, spacelike and orthogonal to K^a. It follows from its definition that L^a is not unique determined by k^A, since any spinor of the form $\tilde{\tau}^A = \tau^A + \lambda k^A$ satisfies $k_A\tilde{\tau}^A = 1$. Hence, L^a transforms as $\tilde{L}^a = L^a + (\lambda + \bar{\lambda})K^a$. Every real value of $\lambda + \bar{\lambda}$ is associated with a family of coplanar vectors L^a, defining a (two-dimensional) half-plane, which is tangent to the null cone along the line defined by K^a (see figure below). This half-plane is the *flagplane* of k^A.

Using this construction, the phase transformation $k^A \to e^{i\theta}k^A$[‡] can be seen to correspond to

[†]This is a definition of a *spin-frame* present in a variety of Penrose works[9].
[‡]This transformation must be accompanied by $\tau \to e^{-i\theta}\tau^A$ to preserve $k_A\tau^A = 1$.

a rotation through 2θ of the flag plane about the flagpole K^a:

$$L^a_{rot} = e^{2i\theta}k^A\bar{\tau}^{A'} + e^{-2i\theta}\tau^A\bar{k}^{A'} = \cos(2\theta)(k^A\bar{\tau}^{A'} + \tau^A\bar{k}^{A'}) + \sin(2\theta)(\underbrace{ik^A\bar{\tau}^{A'} - i\tau^A\bar{k}^{A'}}_{M^a}) = L^a\cos 2\theta + M^a \sin 2\theta$$

Hence, when k^A rotates by θ, the vector L^a rotates by 2θ. Thus, the bivector F^{ab} (composed by the flagpole and flagplane) gives a geometrical representation of a spinor, i.e., the null flag. Note that by (3), there is not anymore phase ambiguity as in (2), which was reduced simply to a sign ambiguity. Therefore, a spinor can be completely realized geometrically only up to sign, which agrees with the arguments presented at the beginning of this section.

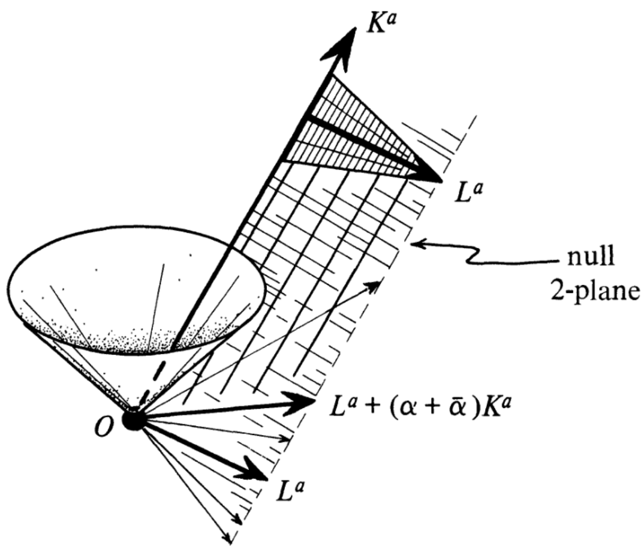

Figure 1 – Geometric picture of a spinor.
Source: PENROSE & RINDLER 1981.

There is a generalization of spinor theory to curved space-time that deals with the Fiber Bundles Theory[10], but this is beyond the scope of this work.

3 Twistor and null lines

The construction of a twistor comes from developing the most fundamental way of describing a null line in space-time. To see this, let $\mathscr{M}$ be the Minkowski space-time. Let L be a null line in $\mathscr{M}$. Let x^i be a set of inertial Cartesian coordinates with origin at O[†]. Let l^i be the position vector of some point P on L. Finally, let n^i be a future-pointing tangent vector to L according to the following picture.

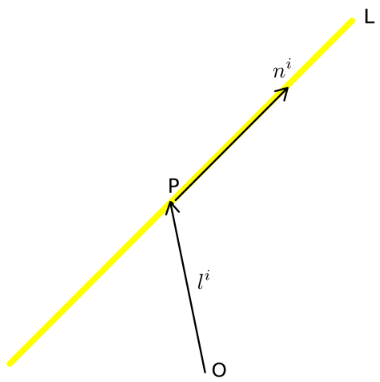

Figure 2 – Representation of a Null Line.
Source: generated by the author.

If one wishes to assign a set of coordinates to the line L, we may do this using the vector n^i and the moment, m^{ij}, related to it, given by the so-called Plucker-Grasmann coordinates

$$m^{ij} = l^i n^j - n^i l^j. \tag{4}$$

Therefore, the ratios of the ten quantities (n^i, m^{ij})[†] will uniquely define L. Nonetheless, there are some constraints for n^i and m^{ij}. Since n^i is tangent to L, it must be null,

$$n^i n_i = 0, \tag{5}$$

which gives one constraint for the ratios. Besides, by the definition,

$$\epsilon_{ijkl} m^{ij} n^k = 0 \Rightarrow \epsilon_{ijkl} m^{ij} m^{kl} = 0, \tag{6}$$

which gives three more constraints.

Therefore, only **five** independent real numbers are left to characterize L. This conclusion makes sense since, for fixed hyperplane $t = cte$, one needs only five parameters to establish a null line in $\mathscr{M}$: three coordinates for the point R and two angles (θ *and* ϕ) to establish the

[†]Here, the metric signature $(+---)$ is used.
[†]Since there are ten quantities, we will only have nine ratios to be considered.

direction of the ray (see figure below) through which it passes. Nonetheless, the study of a null

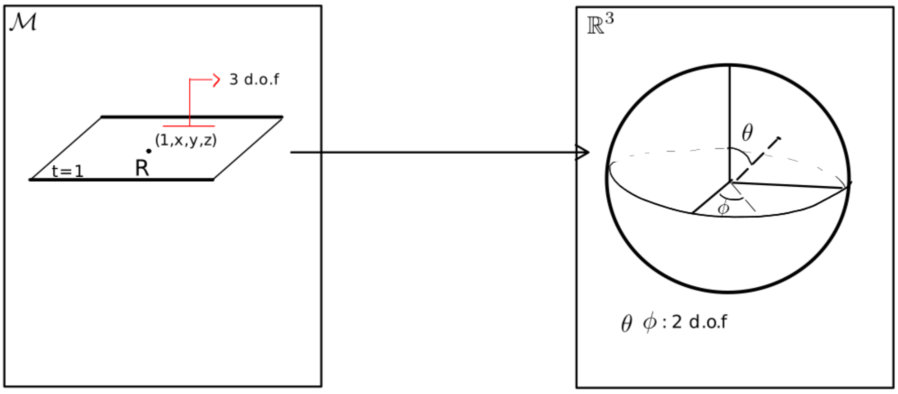

Figure 3 – Characterization of a light-ray.
Source: Generated by the author.

line is simpler if one considers the use of spinor quantities to describe it. The spinor equivalent of the vector n^i is simply given by

$$n^i \leftrightarrow n^{JJ'} = \lambda^J \bar{\lambda}^{J'}. \tag{7}$$

One can think that λ^A is giving the direction of the null line L.

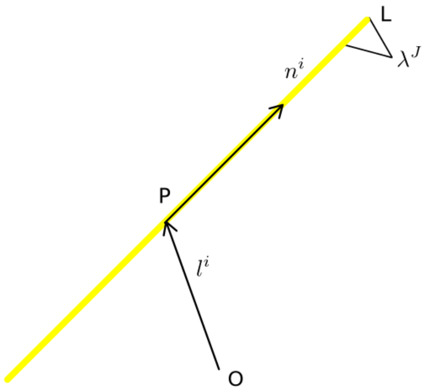

Figure 4 – Spinor representation on a Null Line.
Source: generated by the author.

Besides, one has

$$\begin{aligned} m^{ij} &\leftrightarrow l^{JJ'}\lambda^K\bar{\lambda}^{K'} - \lambda^J\bar{\lambda}^{J'}l^{KK'} \\ &= i\epsilon^{JK}\mu^{(J'}\lambda^{\bar{K}')} - i\mu^{(J}\lambda^{K)}\epsilon^{J'K'}, \end{aligned} \tag{8}$$

where

$$\boxed{\mu_{A'} = -i\lambda^{A} l_{AA'}}\ . \tag{9}$$

Therefore, λ^A and $\mu_{A'}$ together determine l^i and m^{jk}.

Some direct consequences of (9) are the following. If λ^A is multiplied by any factor, then L is unchanged if $\mu_{A'}$ is multiplied by the same factor. Furthermore, $\mu_{A'}$ is unchanged by the choice of P on L. To see this, let $l^{AA'} \to l^{AA'} + a\lambda^A\bar{\lambda}^{A'}$, thus

$$\mu_{A'} = -i\lambda^A(l_{AA'} + a\lambda_A\bar{\lambda}_{A'}) = -i\lambda^A l_{AA'} - ia\cancelto{0}{\lambda^A\lambda_A}\bar{\lambda}_{A'} = -i\lambda^A l_{AA'}.$$

Since the choice of P on L does not matter, one can take a special point of L, namely the intersection of L with the light cone of O.

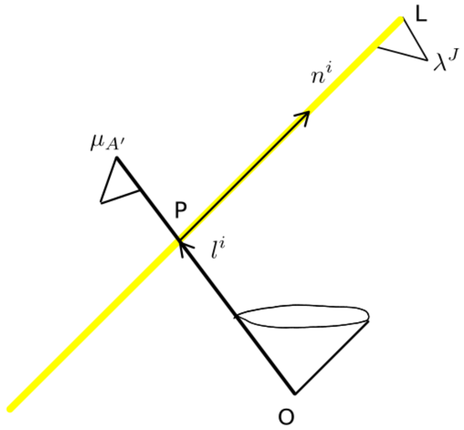

Figure 5 – Complete description of null lines through O and P.
Source: generated by the author.

Thus, the null direction that $\mu_{A'}$ defines is precisely the light ray through O that meets L at P.

In this way, it follows from (9) that

$$l_{AA'} = i(\lambda^B\bar{\mu}_B)^{-1}\bar{\mu}_A\mu_{A'}.^{\dagger} \tag{10}$$

There is still one exceptional case, namely, when $\lambda^B\bar{\mu}_B = 0$, that must be considered. From basic spinorial algebra, this case implies that λ^A and μ^A are proportional to each other. Thus, the null direction of $\mu_{A'}$ is that of L, i.e., L is lying in a null hyperplane through O. To see this, from (9) one would get

$$\bar{\mu}_A = i\bar{\lambda}^{A'} l_{AA'} \Rightarrow \cancelto{0}{\lambda^A\bar{\mu}_A} = i\lambda^A\bar{\lambda}^{A'} l_{AA'} = 0 \iff n^i l_i = 0, \forall\, l^i.$$

Because of the degrees of freedom involved in the description of L and the characterization

[†]This result can be easily checked by looking the consistency of (9) and (10).

of a null line by λ^A, $\mu_{A'}$, the coordinates of a light ray L can be assigned to be the three ratios[‡] of the coordinates of these spinorial quantities. Thus, one may do the following characterization

$$L^0 = \lambda^0,\ L^1 = \lambda^1,\ L^2 = \mu_{0'},\ L^3 = \mu_{1'}, \tag{11}$$

which can be rewritten as

$$\boxed{L^a = (\lambda^A, \mu_{A'})}\ . \tag{12}$$

Since the coordinates of λ^A and $\mu_{A'}$ are complex, the above three ratios correspond to six parameters. Thus, for a complete description of L, one needs a real relation connecting λ^A and $\mu_{A'}$. This expression is obtained by (9) (which will be called, from now on, *incidence relation*). Therefore, to fully describes a light ray in a Minkowski space-time, one must use (12) with (9). Thus, the definition of a **twistor** is a vector (12) with the constraint (9), where

$$l^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \tag{13}$$

and (t, x, y, z) are inertial Cartesian coordinates in $\mathscr{M}$.

3.1 Incidence of Null Lines

The next important thing to do is understand the twistors' geometrical meaning. To do so, one must consider another null line, X, described by the twistor

$$X^a = (\xi^A, \eta_{A'}) \tag{14}$$

with the incidence relation

$$\eta_{A'} = -i\xi^A x_{AA'}, \tag{15}$$

where, in analogy of l^i, x^i is the position of a point in X.

Let's consider that these two null lines intersect each other. Since there is nothing special about l^i and x^i, we could choose $l^i = x^i$. Therefore, the following holds

$$\xi^A \bar{\mu}_A = i\xi^A l_{AA'} \bar{\lambda}^{A'} = -\eta_{A'} \bar{\lambda}^{A'}. \tag{16}$$

Thus, the condition for the two lines to meet is given by (16).

[‡]Since L is null, from its orthogonal relation, one only needs three parameters to a complete description of it.

3.2 Complex Conjugate and Twistor Norm

To be able to continue the study of the incidence of null lines in $\mathscr{M}$, it is necessary to define the *complex conjugate* of a twistor L^a[†] to be $\bar{L}_a$ where

$$\bar{L}_a = (\bar{\mu}_A, \bar{\lambda}^{A'}). \tag{17}$$

In component form,

$$\bar{L}_0 = \bar{\mu}_0,\ \ \bar{L}_1 = \bar{\mu}_1,\ \ \bar{L}_2 = \bar{\lambda}^{0'},\ \ \bar{L}_3 = \bar{\lambda}^{1'}. \tag{18}$$

Using the complex conjugate of a twistor, one can define the *twistor inner product* as

$$X^a \bar{L}_a = \xi^A \bar{\mu}_A + \eta_{A'} \bar{\lambda}^{A'}. \tag{19}$$

From the condition (16) that the two null lines to meet, the expression above becomes

$$X^a \bar{L}_a = 0. \tag{20}$$

Using the incidence relation, the condition for a twistor to satisfies (9), i.e, it represents a null line is

$$L^a \bar{L}_a = 0. \tag{21}$$

It is reasonable to think of $L^a \bar{L}_a = 0$ as a special case of $X^a \bar{L}_a = 0$, since L intersects itself.

Since those lines could intersect at infinity, we adjoin to $\mathscr{M}$ a point at infinity to complete the characterization of null lines.

[†]The complex conjugate of a twistor is regarded as being its dual.

4 Complex Projective Space C

From the previous sections, it was established that the null lines in $\mathscr{M}$ form a system with six degrees of freedom when one includes complexefied null lines (related to Robinson Congruence, see[2]). The members of this system can be given complex coordinates $L^\alpha = (L^0, L^1, L^2, L^3)$ as seen before. An essential feature of the incidence relation discussed earlier was that it is invariant under the transformation $L^\alpha \to \rho L^\alpha$, $\forall \rho \in \mathbb{C}^*$. Thus, only the ratios of these coordinates matter, the only restriction to them being that they all do not vanish. Therefore, the twistor space defines a *complex projective space* which will be denoted here by the letter C.

Therefore, two pictures are equivalent, namely, the one in terms of $\mathscr{M}$ and the one in terms of C. For instance, one may see the object L given by the coordinates listed above either as a null line of $\mathscr{M}$ or as a point of C. Next, it is essential to understand the picture of complex conjugate $L^a \leftrightarrow \bar{L}_a$ in the C picture. From projective geometry, R_a can be viewed as a 2-dimensional *plane R in C*. Therefore, the conjugate relation $L^a \leftrightarrow \bar{L}_a$ defines a relation *point* $\leftrightarrow$ *plane* in the C picture.

The real null lines of $\mathscr{M}$ defines a five-real-dimensional subset, namely $\mathscr{N}$, of C, defined by the condition $L^a \bar{L}_a = 0$. Furthermore, the twistor inner product allows one to define two other subsets of C, namely C^+, defined by $X^a \bar{X}_a > 0$, and C^-, defined by $X^a \bar{X}_a < 0$. These two subsets are disconnected, with $\mathscr{N}$ being the boundary.

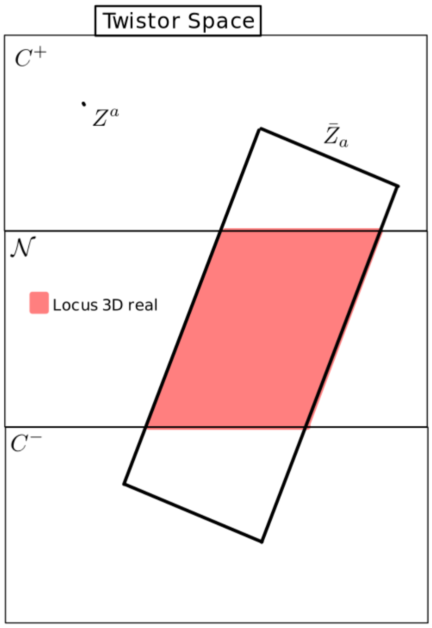

Figure 6 – The C picture.
Source: generated by the author.

Finally, it is essential to know how a point of $\mathscr{M}$ is represented in terms of the C picture. Any point $p \in \mathscr{M}$ can be uniquely represented by two null lines in $\mathscr{M}$, namely by the generators of the light-cone of p. Thus, we need the two generators, K and L, to meet each other (defining the point p). In this way, the generators of p are the null lines common to both $\bar{K}$ and $\bar{L}$, which is a system of rays on $\mathscr{N}$ which lies in the intersection of these polar planes (this intersection being simply a complex projective line in C). Therefore, it follows that any point of $\mathscr{M}$ is represented in the C picture by a complex projective line that lies entirely on $\mathscr{N}$.

This last result can be seen algebraically using the incidence relation. It is convenient to rewrite (9) using the matrix form of it. If Z^a is some twistor in the complex projective space with coordinates $Z^a = (Z^0, Z^1, Z^2, Z^3)$ and l^i is written in Cartesian coordinates, using (13), the incidence relation becomes

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}, \tag{22}$$

Let l^i be fixed; thus

$$\begin{aligned} Z^0 &= \alpha Z^2 + \beta Z^3, \\ Z^1 &= \gamma Z^2 + \sigma Z^3, \end{aligned} \tag{23}$$

where $\alpha, \beta, \gamma, \sigma \in \mathbb{C}$.

The above relations define planes in C. Since both relations must be satisfied, the point $p \in \mathscr{M}$ is determined by the intersection of these planes, i.e., by the projective line that lies in their intersection. Therefore, a point on $\mathscr{M}$ corresponds to a projective line that lies entirely on $\mathscr{N}$.

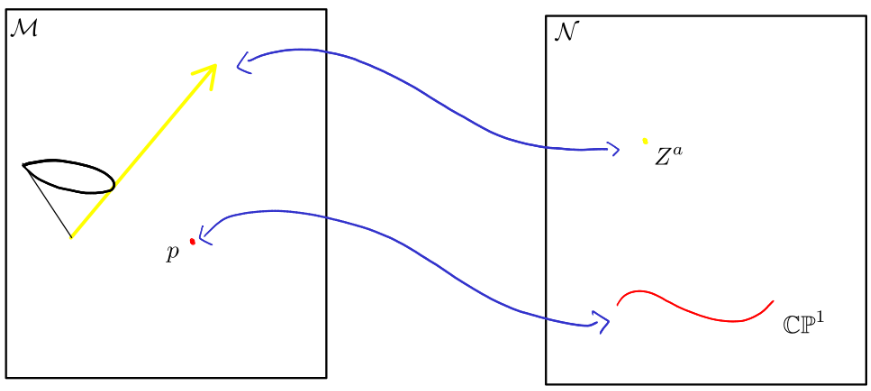

Figure 7 – Non-local correspondence between $\mathscr{M}$ and $\mathscr{N}$.
Source: generated by the author.

5 Complexified Minkowski Space-time

A generic description of twistors can be obtained by complexifying space-time. The idea is to pass to a complex Minkowski space-time and obtain the relations between this and the twistor space. This development will be crucial to the non-linear graviton construction later on.

Let $\mathscr{M}$ be the Minkowski space-time with line element $ds^2 = g_{ab}(x)dx^a dx^b$, where $g_{ab}(x)$ indicates a possible dependence between the components $g_{\mu\nu}$ of g_{ab} and the coordinates adopted. Then, the complexification of $(\mathscr{M}, g_{ab})$ is given when one allows complex values of the coordinates x^μ while extending $g_{ab}(x)$ holomorphically. In this way, a complexified space-time $\mathbb{M}_{\mathbb{C}}$ is defined.

Let us use inertial Cartesian coordinates to explore $\mathbb{M}_{\mathbb{C}}$. Thus, one has $g_{\mu\nu} = diag(-1, 1, 1, 1)$ as in $\mathscr{M}$.

In this "new" space-time, the incidence relation (9) is still the same, but the vector l^a can now have complex components. Let $Z^a = (w^A, \pi_{A'})$ be a null twistor. In this way, the incidence relation assumes

$$w^A = ix^{AA'}\pi_{A'}, \tag{24}$$

where $x^a \in \mathbb{M}_{\mathbb{C}}$. The dual twistor incidence relation is given by

$$\bar{\omega}^{A'} = -i(x^{AA'})^\dagger \bar{\pi}_A. \tag{25}$$

From the discussion made in the section 3, the most general solution of (9) is

$$x^{AA'} = x_0^{AA'} + \lambda^A \pi^{A'}, \tag{26}$$

where $x_0^{AA'}$ is a point that is in incidence with $(\omega^A, \pi_{A'})$.

Therefore, the locus of Z^a in $\mathbb{M}_{\mathbb{C}}$ is a 2-plane generated by all the final points of the vector that starts in $x_0^{AA'}$ and ends in the null vector $\lambda^A \pi^{A'}$. Furthermore, since $\pi^{A'}$ is fixed, the different points of this 2-plane can be obtained by varying λ^A. In this way, all vectors in this plane are orthogonal to each other.

Therefore, the twistor Z^a corresponds to a bi-dimensional plane in $\mathbb{M}_{\mathbb{C}}$, called $\alpha - plane$, every tangent of which has the form $\lambda^A \pi^{A'}$. The same discussion can be made for a dual twistor, which gives us a correspondence of $\bar{Z}_a$ with a 2-plane in $\mathbb{M}_{\mathbb{C}}$, called $\beta - plane$.

5.1 Lorentzian signature

Suppose now that one wants to get back the Lorentzian real Minkowski space-time $\mathscr{M}$ out of $\mathbb{M}_{\mathbb{C}}$. To do this, one lets the usual coordinates (x^0, x^1, x^2, x^3) be real-valued. In terms of the matrix $x^{AA'}$, this condition corresponds to the *hermiticity* of it: $x^{AA'} = (x^{AA'})^\dagger$.

To understand (or recover) the correspondence of twistors and points in space-time, let X $\in C$ be a line. Let $Z \in X$ be any point lying in X. From the incidence relation, it follows that

$$Z^a \bar{Z}_a = ix^{AA'}\lambda_A\bar{\lambda}_{A'} - i(x^{AA'})^{\dagger}\bar{\lambda}_{A'}\lambda_A = i(x - x^{\dagger})^{AA'}\lambda_A\bar{\lambda}_{A'}, \tag{27}$$

but from the previous discussion, $x \in \mathscr{M}$ iff $x^{AA'} = (x^{AA'})^{\dagger}$. Therefore, any line corresponding to a point in $\mathscr{M}$ must be contained in the subset $\mathscr{N}$ of C. In other words, $\mathscr{N}$ is the twistor space associated with $\mathscr{M}$.

Recall that a twistor corresponds to a $\alpha - plane$ in $\mathbb{M}_{\mathbb{C}}$. From the condition $Z^a\bar{Z}_a = 0$, one gets a single tangent vector of this plane, namely $\lambda^A\bar{\lambda}^{A'}$. This result follows from the last expression of (27), without imposing the hermicity of $x^{AA'}$. Therefore, for $Z^a\bar{Z}_a = 0$, one simply needs to have $(x - x^{\dagger})^{AA'} \propto \lambda^A\bar{\lambda}^{A'}$. Thus, we singled out only one tangent vector from the $\alpha - plane$. Therefore, a point $Z^a \in \mathscr{N}$ corresponds to a unique real null geodesic in $\mathscr{M}$. The way of visualizing this is that the geodesic is where the $\alpha - plane$ intersects the real slice $\mathscr{M} \subset \mathbb{M}_{\mathbb{C}}$ (see figure below).

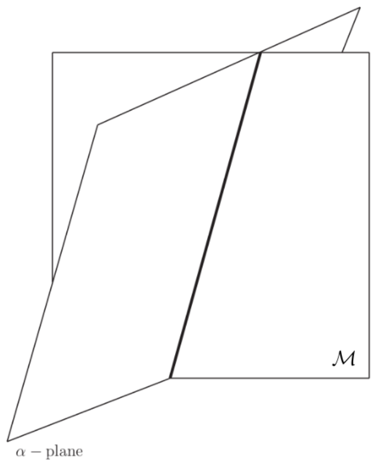

Figure 8 – Intersection between real space-time $\mathscr{M}$ and the $\alpha - plane$ defined by the twistor Z^a.
Source: generated by the author.

6 The Nonlinear Graviton

The Nonlinear Graviton construction[11] is one of the most important results from twistor theory. This framework shows how to construct a Ricci-flat metric from deformations of twistor spaces.

6.1 Curvature Spinors and Right-Flat Space-Times

First, it is important to consider the decomposition of the curvature tensor of space-time in terms of its irreducible spinor components[9]:

$$\begin{aligned} R_{abcd} =& \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \epsilon_{AB}\Phi_{CDA'B'}\epsilon_{C'D'} \\ & + \epsilon_{A'B'}\Phi_{ABC'D'}\epsilon_{CD} + 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}), \end{aligned} \tag{28}$$

where the curvature spinors $\Psi_{ABCD}, \Phi_{ABC'D'}$ and Λ satisfy[9]

$$\Psi_{ABCD} = \Psi_{(ABCD)},\ \Phi_{ABC'D'} = \bar{\Phi}_{ABC'D'} = \Phi_{(AB)(C'D')},\ \Lambda = \bar{\Lambda} \tag{29}$$

and Ψ_{ABCD} and $\bar{\Psi}_{A'B'C'D'}$ correspond, respectively, to the anti-self-dual (a.s.d) and self-dual (s.d) parts of the Weyl curvature tensor,

$$C_{abcd} = C^{-}_{abcd} + C^{+}_{abcd},$$

with

$$C^{-}_{abcd} = \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} \ and \ C^{+}_{abcd} = \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

where

$$\frac{1}{2}\epsilon_{ab}{}^{pq}C_{pqcd} = \pm i C^{\pm}_{abcd}\ .$$

A complex space-time is said to be **right-flat** if $\bar{\Psi}_{A'B'C'D'} = 0$ and $R_{ab} = 0$.

6.2 The existence of α-planes and Anti-Self-Dual curvatures

The next step is to consider the problem of finding α-planes in a complex space-time $\mathbb{M}_{\mathbb{C}}$ with conformal curvature. Two tangents X^a and Y^a to an α-plane necessarily have the form

$$X^a = \lambda^A\pi^{A'},\ Y^a = \mu^A\pi^{A'} \tag{30}$$

for some $\pi^{A'}$. Since they are generating the surface, Frobenius' theorem imposes that the Lie bracket of X^a and Y^a must be some linear combination of these vectors,

$$X^b \nabla_b Y^a - Y^b \nabla_b X^a = \alpha X^a + \beta Y^a, \tag{31}$$

where $\alpha, \beta \in \mathbb{C}^*$.

From (30) it follows that the RHS of (31) has the form $\nu^A \pi^{A'}$, for some ν^A. Therefore, (31) assumes

$$\pi^{A'} \left(X^b \nabla_b Y_a - Y^b \nabla_b X_a \right) = 0.$$

Substituting (30) leads to

$$\pi^{A'} \pi^{B'} \nabla_{BB'} \pi_{A'} = 0. \tag{32}$$

Applying $\pi^{C'} \nabla^B_{C'}$ and using the Spinor Ricci identities[9] leads to

$$\bar{\Psi}_{A'B'C'D'} \pi^{A'} \pi^{B'} \pi^{C'} \pi^{D'} = 0. \tag{33}$$

Therefore, $\bar{\Psi}_{A'B'C'D'} = 0$ is a necessary and sufficient condition that for every point of $\mathbb{M}_{\mathbb{C}}$ and every spinor $\pi^{A'}$ there exists an α-plane through this point of $\mathbb{M}_{\mathbb{C}}$[*]. Therefore, the existence of α-planes in a complex space-time implies that $\mathbb{M}_{\mathbb{C}}$ has an a.s.d curvature (metric).

6.3 Deformation Theory

The last subsection argued that there is a relation between the existence of twistors spaces and the curvature of a complex (but physical, i.e., that admits a Lorentzian signature) space-time $\mathbb{M}_{\mathbb{C}}$. Roger Penrose showed that right-flat metrics in some physical region $\mathscr{R} \subset \mathbb{M}_{\mathbb{C}}$ corresponds to deformations of the corresponding region $\mathscr{U}$ of $\mathbf{T}$, preserving some basic structure (see Refs.[11–13] and their references), where $\mathbf{T}$ is the non-projective twistor space. The deformations of $\mathscr{U}$ may be described as follows. Let U and $\tilde{U}$ be subsets of $\mathscr{U}$. Let $(\omega^A, \pi_{A'})$ and $(\tilde{\omega}^A, \tilde{\pi}_{A'})$, respectively, denote the coordinates of U and $\tilde{U}$. The relation between them is given by a deformation function f^A through an infinitesimal deformation

$$\begin{aligned} \tilde{\omega}^A &= \omega^A + t f^A(\omega^B, \pi_{B'}), \\ \tilde{\pi}_{A'} &= \pi_{A'}, \end{aligned} \tag{34}$$

where t is a deformation parameter and $f^A = \epsilon^{AB} \frac{\partial f}{\partial \omega^B}$ (see Ref.[12] and its references). The class of patching functions can be found in Ref.[14].

[*]Note that in a complex space-time $\bar{\Psi}_{A'B'C'D'}$ is no longer the complex conjugate of Ψ_{ABCD}.

Therefore, the finite deformation is given by solving

$$\frac{d\omega^A}{dt} = -\epsilon^{AB}\frac{\partial f}{\partial \omega^B} \tag{35}$$

with $\omega^A(0) = \omega^A$ and then set $\tilde{\omega}^A = \omega^A(t)$.

6.4 Construction of the metric

Next, there is an explicit way to construct the metric tensor using f^A. A vector $U = U^a\nabla_a$ at some point of $\mathscr{R}$, represented by some displacement $\delta\omega^A$ in $\mathscr{U}$ †, is null iff

$$\delta\omega^A = 0, \tag{36}$$

from where it follows the conformal structure of the metric g. To recover the conformal factor, one needs to consider two null vectors, U and V, at some point $x \in \mathscr{R}$, that vanish at some $\alpha^{A'}$ and $\beta^{A'}$. In the $\mathscr{U}$ picture, they correspond to two displacements $\delta\omega^A$ and $\Delta\omega^A$. Let $\mathscr{S}_\pi(U,V)$ be a skew-symmetric form given by

$$\mathscr{S}_\pi(U,V) := \delta\omega_A\Delta\omega^A. \tag{37}$$

Therefore, the metric g_{ab} is defined by[11,14]

$$g_{ab}U^aV^b := \frac{(\alpha^{A'}\beta_{A'})\mathscr{S}_\pi(U,V)}{(\alpha^{B'}\pi_{B'})(\beta^{C'}\pi_{C'})}. \tag{38}$$

Hence, once established the deformation function of the twistor space, one can construct the corresponding metric of the space-time.

Penrose believes that the spaces generated by such deformations should form the one-particle states of a future quantum theory of gravity[11].

As an example, consider the deformation function

$$f = \frac{(\omega^0)^4}{4\pi_{0'}\pi_{1'}}, \tag{39}$$

defined with respect of two sets of C^+:

$$U = \{Z^a, \pi_{0'} \neq 0\}\,,\ \tilde{U} = \{Z^a, \pi_{1'} \neq 0\}\,.$$

† This relation is regarded to the cross-sections of the fiber bundle of $\mathscr{N}$.

Therefore, (35) becomes

$$\frac{d\omega^0}{dt} = 0; \ \frac{d\omega^1}{dt} = \frac{(\omega^0)^3}{\pi_{0'}\pi_{1'}}. \tag{40}$$

Hence,

$$\tilde{\omega}^0 = \omega^0; \ \tilde{\omega}^1 = \omega^1 + t\frac{(\omega^0)^3}{\pi_{0'}\pi_{1'}}. \tag{41}$$

Thus, ω^0 defines a global function on $\mathbb{CP}^1$, so by the Liouville theorem it must be linear[15]:

$$\omega^0 = w\pi_{0'} + x\pi_{1'}, \tag{42}$$

for some $w, x \in \mathbb{C}$. Hence,

$$\tilde{\omega}^1 - \frac{t}{\pi_{1'}}\left(\omega^3(\pi_{0'}^2) + 3w^2 x\pi_{0'}\pi_{1'} + 3wx^2(\pi_{1'})^2\right) = \omega^1 + \frac{t}{\pi_{0'}}x^3(\pi_{1'})^2 \ .$$

Again, by the Liouville theorem,

$$w^1 = -tx^3\frac{(\pi_{1'})^2}{\pi_{0'}} + y\pi_{0'} + z\pi_{1'} \tag{43}$$

for some $y, z \in \mathbb{C}$. Thus, (42) and (43) give the holomorphic curves in terms of four parameters (w, x, y, z).

From (36) it follows that a vector $\delta\omega^A$ will be null iff the equations

$$\begin{aligned} \delta\omega^0 &= \pi_{0'}\delta w + \pi_{1'}\delta x = 0, \\ \delta\omega^1 &= \pi_{0'}\delta y + \pi_{1'}\delta z - 3tx^2\frac{(\pi_{1'})^2}{\pi_{0'}}\delta x = 0 \end{aligned} \tag{44}$$

have a simultaneous solution. The condition for this is

$$-\delta x\delta y + \delta z\delta w + 3tx^2(\delta w)^2 = 0. \tag{45}$$

Therefore, the conformal metric is

$$g \propto dzdw - dxdy + 3tx^2 dw^2 \ . \tag{46}$$

Finally, to obtain the conformal factor, consider two null vectors U^a and V^a given, for simplicity, by

$$V^a = (\delta x, 0, 0, 0), \ U^a = (0, \Delta y, 0, 0) \ . \tag{47}$$

Correspondingly,

$$\begin{aligned}
&\delta\omega^0 = \pi_{1'}\delta x;\ \ \delta\omega^1 = -3t\frac{(\pi_{1'})^2}{\pi_{0'}}x^2\delta x, \\
&\Delta\omega^0 = 0;\ \ \Delta\omega^1 = \pi_{0'}\delta y \ .
\end{aligned} \tag{48}$$

Next, let $\alpha^{A'} = (1, 0)$ and $\beta^{A'} = (0, 1)$. Hence, (38) becomes

$$g(U, V) = \frac{\mathscr{S}_\pi(U, V)}{\pi_{1'}\pi_{0'}} \ .$$

From (37) and (48) follow

$$\mathscr{S}_\pi = \delta\omega^0\Delta\omega^1 - \delta\omega^1\cancelto{0}{\Delta\omega^0} = \pi_{1'}\pi_{0'}\delta x\delta y \ . \tag{49}$$

Thus,

$$g(U, V) = \delta x\delta y \Rightarrow g_{01} = 1 \ .$$

Therefore, the deformation (39) generates the metric

$$\boxed{g = 2dxdy - 2dwdz - 6tx^2dw^2} \ . \tag{50}$$

6.5 Einstein's Vacuum Equations

In this last subsection, it was shown that the metric (50) is, indeed, a.s.d. The Einstein's equations on vacuum are given by

$$R_{ab} - \frac{1}{2}g_{ab}R = 0, \tag{51}$$

where $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\gamma} - \partial_\delta\Gamma^\alpha_{\beta\gamma} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta}, \tag{52}$$

with

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right). \tag{53}$$

The inverse metric of (50) is

$$g^{\mu\nu} = \begin{pmatrix} 6tx^2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} . \tag{54}$$

Therefore, the only non-vanishing Christoffel symbols are

$$\Gamma^2_{33} = 6tx; \;\; \Gamma^0_{13} = \Gamma^0_{31} = 6tx. \tag{55}$$

Hence, the only non-vanishing components of the Riemann tensor (52) are

$$R^0{}_{131} = -6; \;\; R^2{}_{331} = -6. \tag{56}$$

Thus, it follows that $R_{ab} = 0 \Rightarrow R = 0$. Hence, the metric satisfies (51). Besides, it follows that $C_{abcd} = R_{abcd}$. Then, $C^+_{abcd} = 0 \Rightarrow \bar{\Psi}_{A'B'C'D'} = 0$. Therefore, the metric (50) indeed satisfies the Einstein's vacuum equations and it is right-flat.

7 Conclusion

From this work, one can conclude that it is possible to describe fundamental space-time structures, such as null lines and events, using a complex projective space (the twistor space) related to $\mathscr{M}$ by one central equation. Thus, space-time is no longer a primordial manifold but an emergent one.

Besides, it was shown that the condition for the existence of twistors in space-time leads to some constraints in the curvature of its metric. Then, using holomorphic deformations of two open subsets of the set of null twistors is possible to establish the metric of such space-time, which is Ricci-flat.

The aim of the present work was to present the formalism of twistors and one important physical application of it, without going deep into the applications mentioned in section 1 due to the lack of time and space.

REFERENCES

1 WITTEN, L. Invariants of general relativity and the classification of spaces. *Physical Review,* v.113, p. 357–362, 1959. DOI: 10.1103/PhysRev.113.357.

2 PENROSE, R. Twistor algebra. *Journal of Mathematical Physics, v.* 8, p. 345–366, 1967. DOI: 10. 1016/0370-1573(73)90008-2.

3 OSMAN, Y.; BASHEER, M.; ABDELHALEEM, T. On the twistor method for treating differential equations. *Journal of Mathematics Research*, v. 9, p. 115, 2017. DOI: 10.5539/ jmr. v92n2p115.

4 DUNAJSKI, M. Twistor theory and differential equations. *Journal of Physics A*: mathematical and theoretical, v. 42, 2009. DOI: 10. 1088/1751-8113/42/40/404004.

5 PENROSE, R.; MACCALLUM, M. Twistor theory: an approach to the quantisation of fields and space-time. *Physics Reports*, v. 6, p. 241–315, 1973. DOI: 10.1016/0370-1573(73)90008-2.

6 ATIYAH, M.; DUNAJSKI, M.; MASON, L. Twistor theory at fifty: from contour integrals to twistor strings. *Proceedings of the Royal Society A*: mathematical, physical and engineering science, v. 473, 2017. DOI: 10. 1098/ rspa. 2017. 0530.

7 FRANÇOIS, J. Dilaton from tractor and matter field from twistor. *Journal of High Energy Physics*, v. 6, n. 18, p. 1-17, 2019. DOI: 10.1007/JHEP06(2019)018.

8 DUNAJSKI, M.; PENROSE, R. Quantum state reduction, and Newtonian twistor theory. 2022. Available from: https://arxiv.org/pdf/2203.08567.pdf. Accessible at: 23 Jan. 2022.

9 PENROSE, R.; RINDLER, W. *Spinors and space-time*: two-spinor calculus and relativistic fields. Cambridge: Cambridge University Press, 1984.

10 WALD, R. M. *General relativity*. Chicago: Chicago Universe Press, 1984.

11 PENROSE, R. Nonlinear gravitons and curved twistor theory. *General Relativity Gravitation*, v. 7, p. 31–52, 1976. DOI: 10.1007/BF00762011.

12 HUGGETT, S. A.; TOD, K. P. *An introduction to twistor theory.* 2nd ed. Cambridge: Cambridge University Press, 1994.

13 WARD, R. S.; WELLS, R. O. *Twistor geometry and field theory*. Cambridge: Cambridge University Press, 1991.

14 WARD, R. S. A class of self-dual solutions of Einstein’s equations. *Proceedings of the Royal Society of London Series A*: mathematical and physical sciences, v. 363, n. 1713, p. 289–295, 1978.

15 DUNAJSKI, M. *Solitons, instantons, and twistors*. Oxford: Oxford University Press, 2010. (Oxford graduate texts in mathematics, v. 19)